I: Several CF algorithms

Generalised CFs

· Defn.: A generalised CF (GCF) is

$$(*)\quad [d_0/e_0;\ e_1/d_1,\ e_2/d_2,\ \dots] = \frac{d_0}{e_0} + \cfrac{e_1/e_0}{d_1 + \cfrac{e_2}{d_2 + \cfrac{e_3}{d_3 + \dots}}},$$

$e_n, d_n \in \mathbb{Z}$, $e_n \neq 0$. Its convergents are

$$\frac{p_n}{Q_n} = [d_0/e_0;\ e_1/d_1, \dots, e_n/d_n] \in \mathbb{Q} \cup \{\infty\}.$$

Call $(*)$ a GCF-expansion of $x \in \mathbb{R}$ if

$$x = \lim_n p_n/Q_n.$$

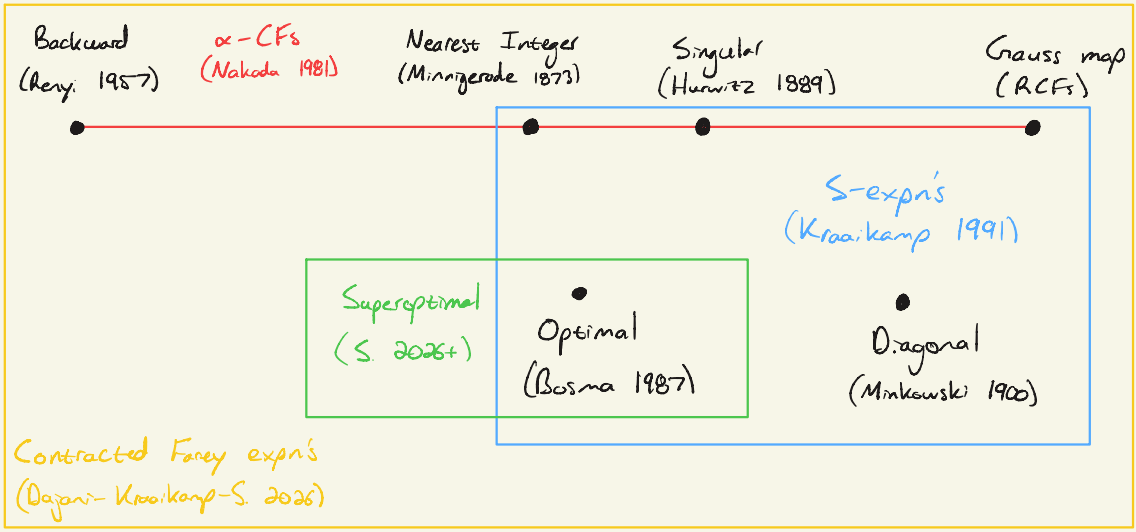
(· Remarks: · Not every GCF is a GCF-expn's.
· Each $x \in \mathbb{R}$ has many GCF-expn's.)

· Eg.: Regular CFs (RCFs) are GCFs with $e_n = 1$, $d_n > 0$ for $n > 0$. Use special notation $a_n := d_n$,

$$[a_0; a_1, a_2, \dots] := [a_0/1;\ 1/a_1,\ 1/a_2, \dots],$$

$$\frac{p_n}{q_n} := \frac{p_n}{Q_n} \quad (\text{reduced},\ p_n, q_n \geq 0)$$

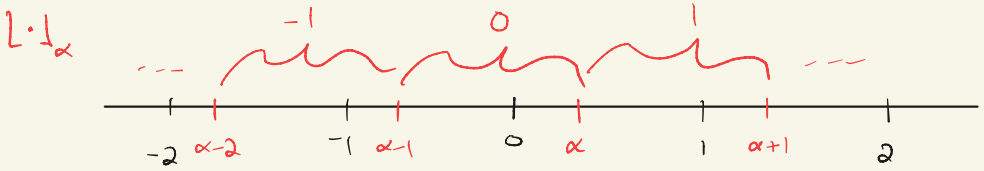
- Several algorithms produce GCF-expn's:



Nakada's α-CFs

- For each $\alpha \in [0,1]$, $G_\alpha : [\alpha-1, \alpha) \to [\alpha-1, \alpha)$ is defined by $G_\alpha(0) := 0$ and for $x \neq 0$,

$$G_\alpha(x) := \frac{1}{|x|} - \left\lfloor \frac{1}{|x|} \right\rfloor_\alpha, \qquad \lfloor y \rfloor_\alpha := \lfloor y + 1 - \alpha \rfloor$$



- Remark: $\alpha = 1$ gives Gauss map $G = G_1$.

- For $x \in \mathbb{R}$, let $x_0^\alpha = x - \lfloor x \rfloor_\alpha$, $x_n^\alpha = G_\alpha^n(x_0^\alpha)$, $n \geq 0$.

- The α-CF-expn. of x is

$$x = \left[d_0^\alpha / 1 \; ; \; e_1^\alpha / d_1^\alpha, \; e_2^\alpha / d_2^\alpha, \ldots \right],$$

where $d_0^\alpha = \lfloor x \rfloor_\alpha$,

$$e_{n+1}^\alpha = \operatorname{sgn}(x_n^\alpha), \qquad d_{n+1}^\alpha = \left\lfloor \frac{1}{|x_n^\alpha|} \right\rfloor_\alpha .$$

- Fact: Each G_α is ergodic w/r/t a (unique up to scalar) σ-finite, absolutely continuous invariant measure (a.c.i.m.) ν_α.

- Nakada (1981) determined ν_α, $\alpha \geq 1/2$, via the natural extension of G_α.

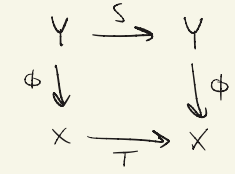
Natural extensions

- Defn.: $(X, \mathcal{B}_X, \mu, T)$ is a factor of $(Y, \mathcal{B}_Y, \nu, S)$ if (mod. null sets) $\exists\, \phi: Y \to X$ surjective, measurable, measure-preserving s.t. $\phi \circ S = T \circ \phi$.

Further, if T non-invertible, S invertible, and

$$\mathcal{B}_Y = \bigvee_{n=0}^{\infty} S^n(\phi^{-1}\mathcal{B}_X),$$

S is a natural extension (NE) of T.



- Facts:
  - NEs always exist, are unique up to isomorphism
  - T ergodic iff S ergodic
  - For $\mu, \nu < \infty$, metric entropies $h_\mu(T)$, $h_\nu(S)$ coincide

- E.g.: Let $([0,1], \mathcal{B}, \nu_G, G)$ be Gauss map. ($\nu_G = \nu_1$, $G = G_1$)

The NE is $(\Omega, \mathcal{B}_\Omega, \overline{\nu}_G, \mathcal{G})$, where

$\Omega = [0,1]^2$; setting $a(x) = \lfloor 1/x \rfloor$,

$$\mathcal{G}(x,y) := \left(\begin{pmatrix} 0 & 1 \\ 1 & a(x) \end{pmatrix}^{-1} \cdot x, \begin{pmatrix} 0 & 1 \\ 1 & a(x) \end{pmatrix}^{T} \cdot y \right) \qquad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau := \frac{a\tau + b}{c\tau + d} \right)$$

$$= \left(\frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \frac{1}{\left\lfloor \frac{1}{x} \right\rfloor + y} \right);$$

and $\overline{\nu}_G$ the a.c.i.m. $\dfrac{d\overline{\nu}_G}{d\lambda} = \dfrac{1}{\log 2} \cdot \dfrac{1}{(1+xy)^2}$.

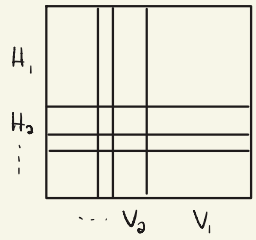
Symbolically,

$$\mathcal{G}([0; a_1, a_2, \ldots], [0; b_1, b_2, \ldots]) = ([0; a_2, a_3, \ldots], [0; a_1, b_1, b_2, \ldots]).$$

Setting $V_k = \{(x,y) \mid a(x) = k\}$,

$H_k = \{(x,y) \mid a(y) = k\}$,

we have $\mathcal{G}(V_k) \overset{\overline{\nu}_G}{=} H_k$.



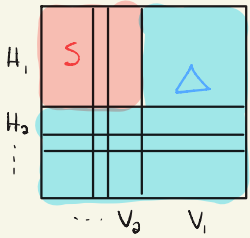
With $\phi(x,y) = x$,

$$\nu_G = \phi_* \overline{\nu}_G \implies \frac{d\nu_G}{d\lambda} = \frac{1}{\log 2} \cdot \frac{1}{1+x}.$$

S-expn's

- Kraaikamp's (1991) S-expn's couple:
  - (i) induced transformations (Kakutani, 1943) of the NE $\mathcal{G}$
  - (ii) singularisation (Lagrange, 1798) of RCFs
- Idea: For $\Delta \subset \Omega$, keep p_{n-1}/q_{n-1} iff $\mathcal{G}^n(x,0) \in \Delta$
  (equiv., remove p_{n-1}/q_{n-1} iff $\mathcal{G}^n(x,0) \in S := \Omega \setminus \Delta$)
- Singularisation is based on

$$a_{n+1} + \cfrac{1}{1 + \cfrac{1}{a_{n+1}+t}} = a_{n-1} + 1 + \cfrac{-1}{a_{n+1}+1+t}$$



- Exercise: Show this 'local surgery' on $x = [0; a_1, a_2, \ldots]$ produces a GCF-expn. of x with convergents $p_0/q_0, \ldots, p_{n-2}/q_{n-2}, p_n/q_n, p_{n+1}/q_{n+1}, \ldots$
- Singularisation can be performed (possibly) i.o., but
  - (i) can only remove p_{n-1}/q_{n-1} if $a_n = 1$
    ($\Rightarrow S \subset H_1$, since $\mathcal{G}^n(x,0) = ([0; a_{n+1}, a_{n+2}, \ldots], [0; a_n, \ldots, a_1])$)
  - (ii) cannot remove both p_{n-1}/q_{n-1}, p_n/q_n
    ($\Rightarrow S \cap \mathcal{G}(S) = \emptyset$)

- Despite restrictions, many GCF-algorithms are S-expns: optimal, diagonal, α-CFs, ...
  but only for $\alpha \geq 1/2$.

The Farey (tent) map

- Define $\varepsilon : [0,1] \to \{0,1\}$,

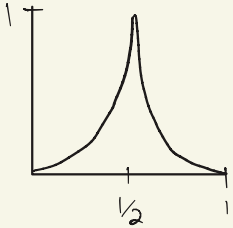
$$\varepsilon(x) := \begin{cases} 0, & x \leq 1/2 \\ 1, & x > 1/2 \end{cases}$$

and

$$A_\varepsilon := \begin{pmatrix} 1-\varepsilon & \varepsilon \\ 1 & 1 \end{pmatrix} \in \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}, \quad \varepsilon \in \{0,1\}.$$

The Farey map $F : [0,1] \to [0,1]$ is

$$F(x) := A_{\varepsilon(x)}^{-1} \cdot x = \begin{cases} \frac{x}{1-x}, & \varepsilon(x) = 0 \\ \frac{1-x}{x}, & \varepsilon(x) = 1 \end{cases}$$



- Fact: F has an ergodic, infinite, σ-finite, a.c.i.m. μ with density $\frac{d\mu}{d\lambda} = \frac{1}{x}$.
- (Symbolic) Dynamics of F are related to G_1:
  if $x = [0; a_1, a_2, \ldots]$, then

$$\frac{1-x}{x} = \frac{1}{x} - 1 = [a_1 - 1; a_2, a_3, \ldots] \implies$$

$$F(x) = \begin{cases} [0; a_1 - 1, a_2, a_3, \dots], & \varepsilon(x) = 0 \quad (a_1 > 1) \\ [0; a_2, a_3, \dots], & \varepsilon(x) = 1 \quad (a_1 = 1) \end{cases}$$

$$\Rightarrow \quad F^{a(x)}(x) = F^{a_1}(x) = G_1(x).$$

- Let $x_n := F^n(x)$, $\varepsilon_{n+1} := \varepsilon(x_n)$. From above,

$$\varepsilon_1 \varepsilon_2 \varepsilon_3 \cdots = O^{a_1 - 1} \mid O^{a_2 - 1} \mid \cdots$$

- Note $\quad x_{n+1} = F(x_n) = A_{\varepsilon_{n+1}}^{-1} \cdot x_n$

$$\Rightarrow \quad x = A_{\varepsilon_1} \cdot x_1 = \underbrace{(A_{\varepsilon_1} \cdots A_{\varepsilon_n})}_{A_{[0,n]}} \cdot x_n \qquad (A_{[0,0]} := \mathrm{Id})$$

- Let $j_n, \lambda_n \in \mathbb{N}$ be s.t.

$$n = a_1 + \cdots + a_{j_n} + \lambda_n, \qquad 0 \le \lambda_n < a_{j_n + 1}.$$

- Then $\quad \varepsilon_1 \cdots \varepsilon_n = O^{a_1 - 1} \mid \cdots O^{a_{j_n} - 1} \mid O^{\lambda_n}$

and $\quad A_0^{a-1} A_1 = \begin{pmatrix} 1 & 0 \\ a-1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$

$$\Rightarrow$$

$$A_{[0,n]} = A_0^{a_1 - 1} A_1 \cdots A_0^{a_{j_n} - 1} A_1 A_0^{\lambda_n}$$

$$= \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_{j_n} \end{pmatrix}}_{\begin{pmatrix} p_{j_n - 1} & p_{j_n} \\ q_{j_n - 1} & q_{j_n} \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ \lambda_n & 1 \end{pmatrix} = \begin{pmatrix} \lambda_n p_{j_n} + p_{j_n - 1} & p_{j_n} \\ \lambda_n q_{j_n} + q_{j_n - 1} & q_{j_n} \end{pmatrix}.$$

- Denoting entries by $\begin{pmatrix} u_n & t_n \\ s_n & r_n \end{pmatrix} := A_{[0,n]}$,

$$\left(\frac{u_n}{s_n}\right)_{n\geq 0} = \left(\frac{p_{-1}}{q_{-1}}, \frac{p_0+p_{-1}}{q_0+q_{-1}}, \ldots, \frac{(a_1-1)p_0+p_{-1}}{(a_1-1)q_0+q_{-1}}, \right.$$

$$\left. \frac{p_0}{q_0}, \frac{p_1+p_0}{q_1+q_0}, \ldots, \frac{(a_2-1)p_1+p_0}{(a_2-1)q_1+q_0}, \ldots \right)$$

produces all RCF-convergents and <u>mediants</u> of x.

<u>Farey expn's</u>

- With defns / notation above, one finds

$$x_n + 1 = 2 - \varepsilon_{n+1} + \frac{(-1)^{\varepsilon_{n+1}+1}}{x_{n+1}+1}$$

- Q: Does

$$(*) \quad x + \cancel{1} = \left[(\cancel{2}^{\,1} - \varepsilon_1)/1 ; \frac{(-1)^{\varepsilon_1+1}}{2-\varepsilon_2}, \frac{(-1)^{\varepsilon_2+1}}{2-\varepsilon_3}, \ldots \right] ?$$

- A: Yes! But $P_n/Q_n \to x$ slowly...

- Prop.: $P_n/Q_n = u_{n+1}/s_{n+1} \quad \forall n$.

(- Proof by induction using defn. of $\begin{pmatrix} u_n & t_n \\ s_n & r_n \end{pmatrix} = A_{[0,n]}$, recurrence relations on P_n, Q_n)

- Defn.: Call $(*)$ the <u>Farey expn.</u> of x, u_n/s_n the <u>Farey convergents</u>.

Ito's NE of F

- Ito (1989) introduced the NE $(\Omega, \mathcal{B}, \bar{\mu}, \mathcal{F})$ of $([0,1], \mathcal{B}, \mu, F)$. Here $\Omega = [0,1]^2$;

$$\mathcal{F}(x,y) := \left(A_{\varepsilon(x)}^{-1} \cdot x,\ A_1 A_{\varepsilon(x)}^{T} A_1^{-1} \cdot y\right)$$

$$= \begin{cases} \left(\frac{x}{1-x},\ \frac{y}{1+y}\right), & \varepsilon(x) = 0, \\ \left(\frac{1-x}{x},\ \frac{1}{1+y}\right), & \varepsilon(x) = 1; \end{cases}$$

and $\bar{\mu}$ is the ergodic, infinite, σ-fin., a.c.i.m. with $\frac{d\bar{\mu}}{d\lambda} = \frac{1}{(x+y-xy)^2}$.

- Symbolically,

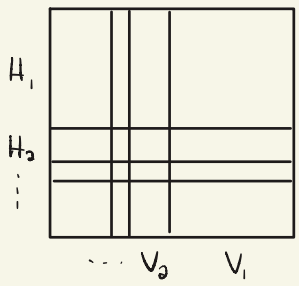
$$\mathcal{F}(z) = \mathcal{F}([0; a_1, a_2, \ldots], [0; b_1, b_2, \ldots])$$

$$= \begin{cases} ([0; a_1 - 1, a_2, \ldots], [0; b_1 + 1, b_2, \ldots]), & \varepsilon(x) = 0 \\ ([0; a_2, a_3, \ldots], [0; a_1, b_1, b_2, \ldots]), & \varepsilon(x) = 1 \end{cases}$$

- Recall

$$n = a_1 + \ldots + a_{j_n} + \lambda_n.$$

One finds (for $b_1 = 1$, $n \geq a_1$),



$$\mathcal{F}^\wedge(z) = \left([0; a_{j_n+1} - \lambda_n, a_{j_n+2}, \dots], \right.$$
$$\left. [0; \lambda_n + 1, a_{j_n}, \dots, a_1, b_2, b_3, \dots] \right)$$

$$\in V_{a_{j_n+1} - \lambda_n} \cap H_{\lambda_n + 1}.$$

- The 1-1 correspondence

$$\mathcal{F}^\wedge(z) \longleftrightarrow \frac{u_n}{s_n}$$

$\Rightarrow$ certain subregions $R \subset \Omega$ 'announce' certain Farey convergents, e.g.:

$$\mathcal{F}^\wedge(z) \in V_{a-\lambda} \cap H_{\lambda+1}$$

$$\Longleftrightarrow$$

$$\frac{u_n}{s_n} = \frac{\lambda p_j + p_{j-1}}{\lambda q_j + q_{j-1}}, \quad 0 \le \lambda < a = a_{j+1} \quad \text{for some } j.$$

- E.g.

$$\mathcal{F}^\wedge(z) \in V_{4-2} \cap H_{2+1}$$

$$\Longleftrightarrow$$

$$\frac{u_n}{s_n} = \frac{2p_j + p_{j-1}}{2q_j + q_{j-1}}, \quad a_{j+1} = 4$$

